Example: Boundary Conditions

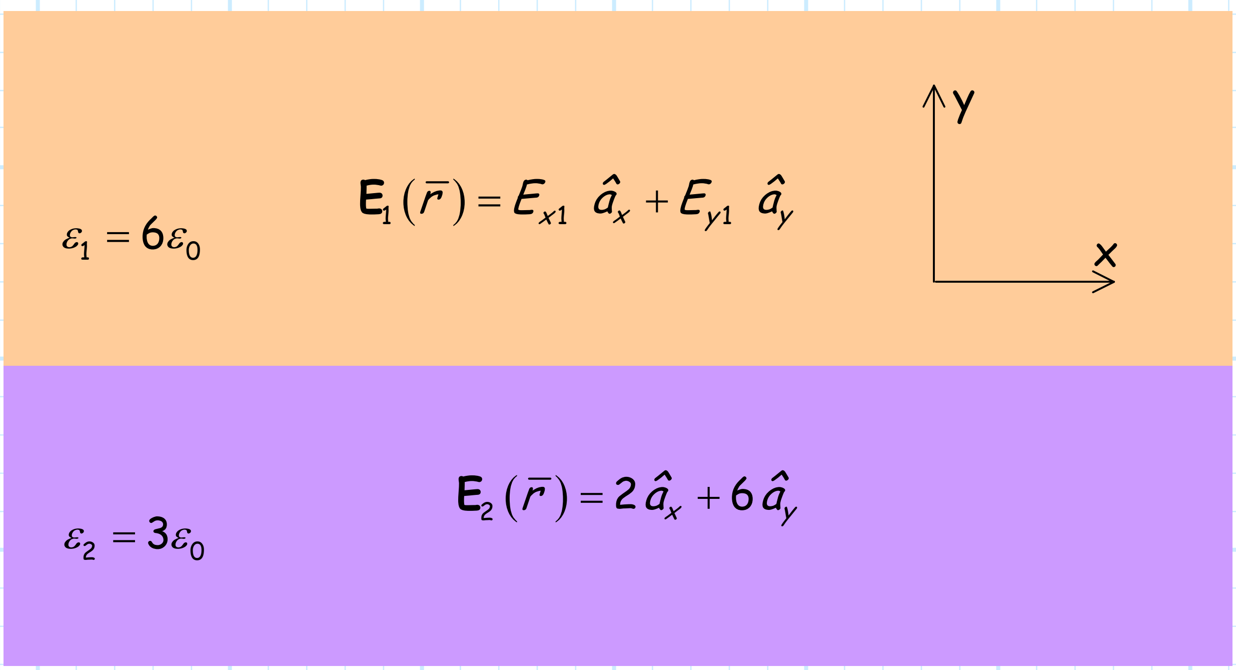
Two slabs of dissimilar **dielectric** material share a common **boundary**, as shown below.

It is known that the electric field in the **lower** dielectric region is:

$$\mathbf{E}_2(\bar{r}) = 2\,\hat{a}_x + 6\,\hat{a}_y \quad \left[V/m \right]$$

and it is known that the electric field in the top region is likewise some **constant** field:

$$\mathbf{E}_1(\bar{r}) = E_{x1}\;\hat{a}_x + E_{y1}\;\hat{a}_y \quad \left[V/m \right]$$



In **each** dielectric region, let's determine (in terms of ε_0):

1) the **electric field**
2) the **electric flux density**
3) the **bound *volume* charge density** (i.e., the equivalent polarization charge density) within the dielectric.
4) the **bound *surface* charge density** (i.e., the equivalent polarization charge density) at the dielectric interface

Since we already know the electric field in the region, let's evaluate **region 2** first.

We can easily determine the **electric flux density** within the region:

$$\begin{aligned} \mathbf{D}_2(\bar{r}) &= \varepsilon_2 \mathbf{E}_2(\bar{r}) \\ &= 3\,\varepsilon_0 \left(2\,\hat{a}_x + 6\,\hat{a}_y\right) \\ &= 6\varepsilon_0\,\hat{a}_x + 18\varepsilon_0\,\hat{a}_y \quad \left[C/m^2 \right] \end{aligned}$$

Likewise, the polarization vector within the region is:

$$\begin{aligned} \mathbf{P}_2(\bar{r}) &= \varepsilon_0 \chi_{e2}\, \mathbf{E}_2(\bar{r}) \\ &= \varepsilon_0\left(\varepsilon_{r2} - 1\right)\left(2\,\hat{a}_x + 6\,\hat{a}_y\right) \\ &= \varepsilon_0\left(3 - 1\right)\left(2\,\hat{a}_x + 6\,\hat{a}_y\right) \\ &= 4\varepsilon_0\,\hat{a}_x + 12\varepsilon_0\,\hat{a}_y \quad \left[C/m^2 \right] \end{aligned}$$

Q: *Why did we determine the **polarization** vector? It is **not** one of the quantities this problem asked for!*

A: True! But the problem **did** ask for the equivalent **bound charge densities** (both volume and surface) within the dielectric. We need to know polarization vector $\mathbf{P}(\bar{r})$ to find this **bound** charge!

Recall the bound **volume** charge density is:

$$\rho_{vp}(\bar{r}) = -\nabla \cdot \mathbf{P}(\bar{r})$$

and the bound **surface** charge density is:

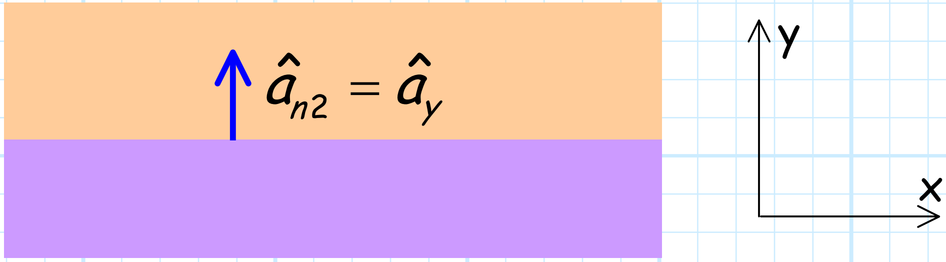
$$\rho_{sp}(\bar{r}) = \mathbf{P}(\bar{r}) \cdot \hat{a}_n$$

Since the polarization vector $\mathbf{P}(\bar{r})$ is a **constant** (i.e., it has precisely the same magnitude and direction at every point with region 2), we find that the divergence of $\mathbf{P}(\bar{r})$ is **zero,** and thus the volume bound charge density is zero within the region:

$$\begin{aligned} \rho_{vp2}(\bar{r}) &= -\nabla \cdot \mathbf{P}_2(\bar{r}) \\ &= -\nabla \cdot \left(4\varepsilon_0\,\hat{a}_x + 12\varepsilon_0\,\hat{a}_y\right) \\ &= 0 \quad \left[{}^{C}\!/\!_{m^3}\right] \end{aligned}$$

However, we find that the **surface** bound charge density is **not** zero!

Note that the unit vector normal to the **surface** of the bottom dielectric slab is $\hat{a}_{n2} = \hat{a}_y$:



Since the polarization vector is constant, we know that its value at the **dielectric interface** is likewise equal to $4\varepsilon_0\,\hat{a}_x + 12\varepsilon_0\,\hat{a}_y$. Thus, the equivalent polarization (i.e., **bound**) **surface charge density** on the top of region 2 (at the dielectric interface) is

$$\begin{aligned}\rho_{sp2}(\bar{r}_b) &= \mathbf{P}_2(\bar{r}_b)\cdot\hat{a}_{n2}\\ &= \left(4\varepsilon_0\,\hat{a}_x + 12\varepsilon_0\,\hat{a}_y\right)\cdot\hat{a}_y\\ &= 12\varepsilon_0 \quad \left[{}^{C}\!/_{m^2}\right]\end{aligned}$$

Now, let's determine these same quantities for **region 1** (i.e., the **top** dielectric slab).

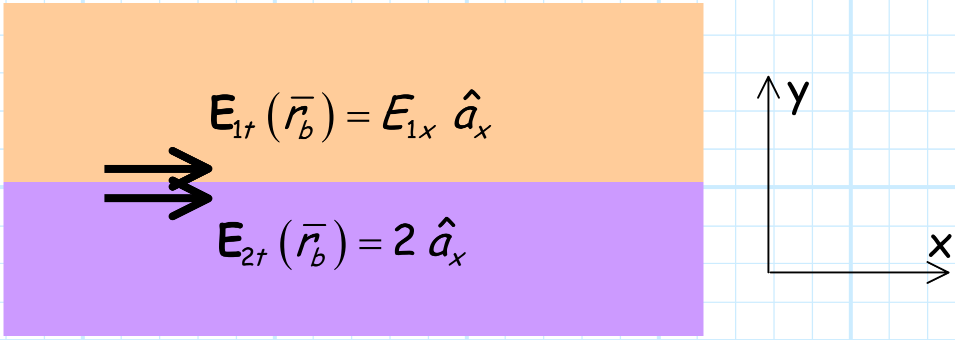
Q1: *How the heck can we do this? We don't know* ***anything*** *about the fields in region 1 !*

A1: True! We don't know $\mathbf{E}_1(\bar{r})$ or $\mathbf{D}_1(\bar{r})$ or even $\mathbf{P}_1(\bar{r})$. However, we know the **next** best thing—we know $\mathbf{E}_2(\bar{r})$ and $\mathbf{D}_2(\bar{r})$ and even $\mathbf{P}_2(\bar{r})$!

Q2: *Huh!?!*

A2: We can use **boundary conditions** to transfer our solutions from region 2 into region 1!

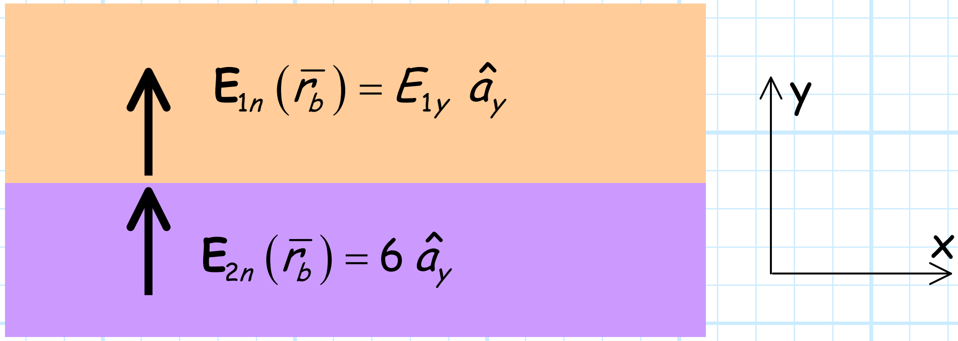
First, we note that **at the dielectric interface**, the vector components of the electric fields **tangential** to the interface are $\mathbf{E}_{1t}(\bar{r}_b) = E_{1x}\ \hat{a}_x$ and $\mathbf{E}_{2t}(\bar{r}_b) = 2\ \hat{a}_x$:



Thus, applying the **boundary condition** $\mathbf{E}_{1t}(\bar{r}_b) = \mathbf{E}_{2t}(\bar{r}_b)$, we find:

$$\begin{aligned}
\mathbf{E}_{1t}(\bar{r}_b) &= \mathbf{E}_{2t}(\bar{r}_b) \\
E_{1x}\ \hat{a}_x &= 2\ \hat{a}_x \\
E_{1x}\ \hat{a}_x \cdot \hat{a}_x &= 2\ \hat{a}_x \cdot \hat{a}_x \\
E_{1x} &= 2
\end{aligned}$$

Likewise, we note that **at the dielectric interface**, the vector components of the electric fields **normal** to the interface are $\mathbf{E}_{1n}(\bar{r}_b) = E_{1y}\ \hat{a}_y$ and $\mathbf{E}_{2n}(\bar{r}_b) = 6\ \hat{a}_y$:



Here, we can apply a **second boundary condition**, $\varepsilon_1 \mathbf{E}_{1n}(\bar{r}_b) = \varepsilon_2 \mathbf{E}_{2n}(\bar{r}_b)$:

$$\varepsilon_1 \mathbf{E}_{1n}(\bar{r}_b) = \varepsilon_2 \mathbf{E}_{2n}(\bar{r}_b)$$

$$6\varepsilon_0\ E_{y1}\ \hat{a}_y = 3\varepsilon_0\ 6\ \hat{a}_y$$

$$E_{y1}\ \hat{a}_y = 3\ \ \hat{a}_y$$

$$E_{y1}\ \ \hat{a}_y \cdot \hat{a}_y = 3\ \ \hat{a}_y \cdot \hat{a}_y$$

$$E_{y1} = 3$$

Thus, we have concluded using boundary conditions that $E_{x1} = 2$ and $E_{y1} = 3$, or the electric field in the top region is:

$$\mathbf{E}_1(\bar{r}) = 2\ \ \hat{a}_x + 3\ \ \hat{a}_y \quad \left[{}^{V}\!/_{m}\right]$$

Likewise, we can find the **electric flux density** by multiplying by the permittivity of region 1 ($\varepsilon_1 = 6\varepsilon_0$):

$$\mathbf{D}_1(\bar{r}) = \varepsilon_1\, \mathbf{E}_1(\bar{r})$$

$$= 12\,\varepsilon_0\ \ \hat{a}_x + 18\,\varepsilon_0\ \ \hat{a}_y \quad \left[{}^{C}\!/_{m^2}\right]$$

Note we could have solved this problem **another** way!

Instead of applying boundary conditions to $\mathbf{E}_2(\bar{r})$, we **could** have applied them to **electric flux density** $\mathbf{D}_2(\bar{r})$:

$$\mathbf{D}_2(\bar{r}) = 6\varepsilon_0\,\hat{a}_x + 18\varepsilon_0\,\hat{a}_y \quad \left[{}^{C}\!/\!{}_{m^2}\right]$$

We know that the **electric flux density** within region 1 must be constant, i.e.:

$$\mathbf{D}_1(\bar{r}) = D_{x1}\,\hat{a}_x + D_{y1}\,\hat{a}_y \quad \left[{}^{C}\!/\!{}_{m^2}\right]$$

and that the vector fields $\mathbf{D}_1(\bar{r})$ and $\mathbf{D}_2(\bar{r})$ **at the interface** are related by the **boundary conditions**:

$$\frac{\mathbf{D}_{1t}(\bar{r}_b)}{\varepsilon_1} = \frac{\mathbf{D}_{2t}(\bar{r}_b)}{\varepsilon_2}$$

and

$$\mathbf{D}_{1n}(\bar{r}_b) = \mathbf{D}_{2n}(\bar{r}_b)$$

It is evident that for this problem:

$$\mathbf{D}_{1t}(\bar{r}_b) = D_{x1}\,\hat{a}_x$$

$$\mathbf{D}_{1n}(\bar{r}_b) = D_{y1}\,\hat{a}_y$$

and for region 2:

$$\mathbf{D}_{2t}(\bar{r}_b) = 12\varepsilon_0\, \hat{a}_x$$

$$\mathbf{D}_{2n}(\bar{r}_b) = 18\varepsilon_0\, \hat{a}_y$$

Combining the results, we find the **two boundary conditions** are:

$$\frac{\mathbf{D}_{1t}(\bar{r}_b)}{\varepsilon_1} = \frac{\mathbf{D}_{2t}(\bar{r}_b)}{\varepsilon_2}$$

$$\frac{D_{1x}\,\hat{a}_x}{6\varepsilon_0} = \frac{6\varepsilon_0\hat{a}_x}{3\varepsilon_0}$$

$$D_{1x}\,\hat{a}_x = 12\varepsilon_0\,\hat{a}_x$$

$$D_{1x}\,\hat{a}_x\cdot\hat{a}_x = 12\varepsilon_0\,\hat{a}_x\cdot\hat{a}_x$$

$$D_{1x} = 12\varepsilon_0$$

and:

$$\mathbf{D}_{1n}(\bar{r}_b) = \mathbf{D}_{2n}(\bar{r}_b)$$

$$D_{1y}\,\hat{a}_y = 18\varepsilon_0\,\hat{a}_y$$

$$D_{1y}\,\hat{a}_y\cdot\hat{a}_y = 18\varepsilon_0\,\hat{a}_y\cdot\hat{a}_y$$

$$D_{1y} = 18\varepsilon_0$$

Therefore, we find that the **electric flux density** is:

$$\mathbf{D}_1(\bar{r}) = 12\varepsilon_0\,\hat{a}_x + 18\varepsilon_0\,\hat{a}_y \quad \left[{}^{C}\!/\!{}_{m^2}\right]$$

Precisely the **same** result as before!

Likewise, we can find the **electric field** in region 1 by dividing by the dielectric permittivity:

$$
\begin{aligned}
\mathbf{E}_1(\bar{r}) &= \frac{\mathbf{D}_1(\bar{r})}{\varepsilon_1} \\
&= \frac{12\varepsilon_0\,\hat{a}_x + 18\varepsilon_0\,\hat{a}_y}{6\varepsilon_0} \\
&= 2\,\hat{a}_x + 3\,\hat{a}_y \quad \left[\,V/m\,\right]
\end{aligned}
$$

Again, the **same** result as before!

Now, finishing this problem, we need to find the **polarization** vector $\mathbf{P}_1(\bar{r})$:

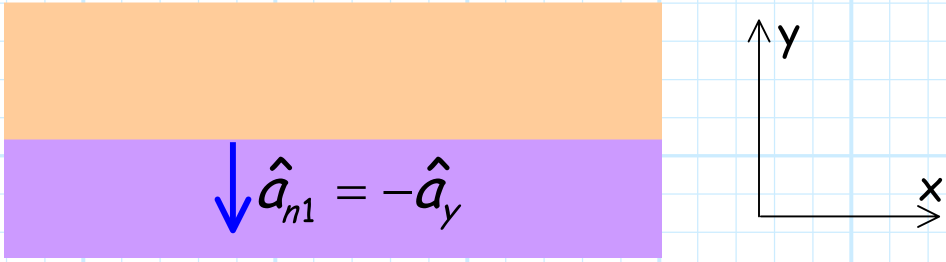
$$
\begin{aligned}
\mathbf{P}_1(\bar{r}) &= \varepsilon_0\left(\varepsilon_{r1} - 1\right)\mathbf{E}_1(\bar{r}) \\
&= \varepsilon_0(6-1)\left(2\,\hat{a}_x + 3\,\hat{a}_y\right) \\
&= 10\,\varepsilon_0\,\hat{a}_x + 15\,\varepsilon_0\,\hat{a}_y \quad \left[\,C/m^2\,\right]
\end{aligned}
$$

Thus, the **volume** charge density of **bound** charge is again **zero**:

$$
\begin{aligned}
\rho_{vp1}(\bar{r}) &= -\nabla \cdot \mathbf{P}_1(\bar{r}) \\
&= -\nabla \cdot \left(10\,\varepsilon_0\,\hat{a}_x + 15\,\varepsilon_0\,\hat{a}_y\right) \\
&= 0
\end{aligned}
$$

However, we again find that the **surface** bound charge density is **not** zero!

Note that the unit vector **normal** to the **bottom** surface of the **top** dielectric slab points **downward**, i.e., $\hat{a}_{n1} = -\hat{a}_y$:



Since the polarization vector is **constant**, we know that its value **at the dielectric interface** is likewise equal to $10\varepsilon_0\,\hat{a}_x + 15\varepsilon_0\,\hat{a}_y$.

Thus, the equivalent polarization (i.e., **bound**) **surface charge density** on the bottom of region 1 (at the dielectric interface) is:

$$\begin{aligned}\rho_{sp1}(\bar{r}_b) &= \mathbf{P}_1(\bar{r}_b)\cdot\hat{a}_{n1}\\ &= \left(10\varepsilon_0\,\hat{a}_x + 15\varepsilon_0\,\hat{a}_y\right)\cdot\left(-\hat{a}_y\right)\\ &= -15\varepsilon_0 \quad \left[{C}/{m^2}\right]\end{aligned}$$

Now, we can determine the **net** surface charge density of **bound** charge that is lying **on the dielectric interface**:

$$\begin{aligned}\rho_{sp}(\bar{r}_b) &= \rho_{sp1}(\bar{r}_b) + \rho_{sp2}(\bar{r}_b)\\ &= -15\varepsilon_0 + 12\varepsilon_0\\ &= -3\varepsilon_0 \quad \left[{C}/{m^2}\right]\end{aligned}$$